CHAPTER # 2 PULSE TRANSFER FUNCTION

<u>After completing this chapter, the students will be able to:</u>

- Obtain the open-loop transfer function for digital control systems,
- Obtain the closed-loop transfer function for digital control systems,

1. Pulse Transfer Function for Open-Loop Systems

In continuous control systems, which represented in S-domain as shown in Fig. 1, the transfer function can be obtained as:

$$Y(s) = R(s).G(s) \rightarrow \frac{Y(s)}{R(s)} = G(s)$$

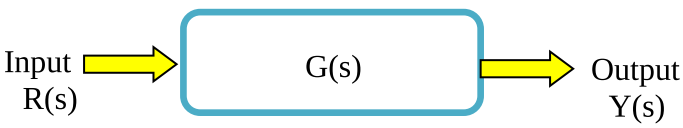


Fig. 1 Analog control system

If the input signal R(s) is sampled, it becomes R(s)* or R(z). Also, If the output signal Y(s) is sampled, it becomes Y(s)* or Y(z). Therefore, the plant G(s) is only defined at sampling instants only, this means it becomes sampled G(s)* or G(z). The block diagram of such system is shown in Fig. (2).

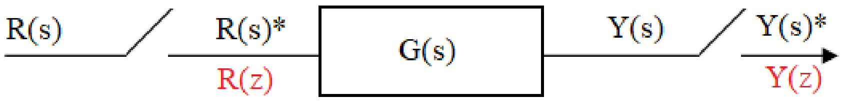


Fig. 2 Digital control system

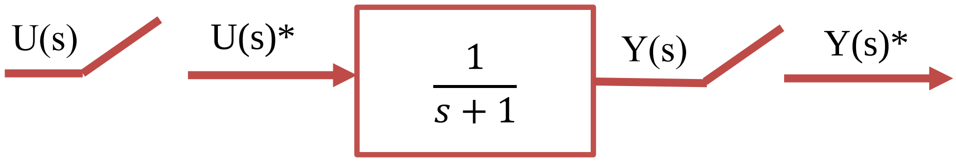
In that case the value of Y(z)/R(z) is called the transfer function of digital control system or Pulse Transfer Function as given below.

$$Y(z) = R(z).G(z) \rightarrow \frac{Y(z)}{R(z)} = G(z)$$

The pulse transfer function is defined as the ratio of the *z*-transform of the output of a linear Time Invariant (LTI) system to the *z*-transform of its input, when the initial conditions are zero.

Example:

A unit step signal is applied to the following system. **Calculate** and **draw** the output response of the system, assuming a sampling period of T = 1 s.



For this system we can write:

$$Y(z) = G(z)u(z)$$

The z-transform of a unit-step function is

$$u(z) = \frac{z}{z-1}$$

the z-transform of $G(s)$ is:

$$G(z) = \mathscr{Z}\{G(s)\} = \mathscr{Z}\left\{\frac{1}{s+1}\right\} = \frac{z}{z-e^{-T}}$$

Thus, the output is given by

$$Y(z) = u(z)G(z) = \frac{z}{z-1}\frac{z}{z-e^{-T}} = \frac{z^2}{(z-1)(z-e^{-T})}$$

Since $T = 1$ sec and $e^{-1} = 0.368$

$$Y(z) = \frac{z^2}{(z-1)(z-0.368)}$$

The output response can be obtained by finding the inverse z-transform of $Y(z)$. Using partial fractions,

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.368} = \frac{1.582}{z-1} - \frac{0.582}{z-0.368}$$

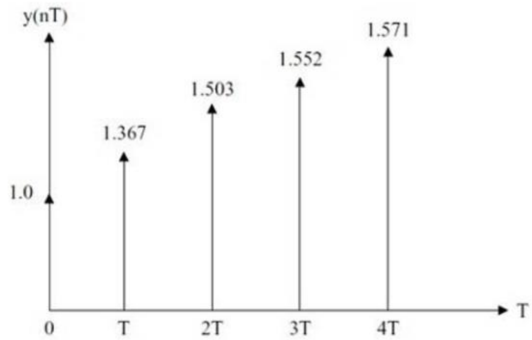
$$Y(z) = \frac{1.582z}{z-1} - \frac{0.582z}{z-0.368}$$

From the z-transform tables we find

$$y(k) = 1.582 - 0.582\,(0.368)^k$$

The first few output samples are:

$$y(0) = 1,$$
$$y(1) = 1.367,$$
$$y(2) = 1.503,$$
$$y(3) = 1.552,$$
$$y(4) = 1.571,$$
$$\vdots$$



It is important to know, if there are samplers between blocks as shown in Fig. 3, then the pulse transfer function of the two blocks can be combined as:

$$\frac{Y(z)}{R(z)} = G_1(z)G_2(z)$$

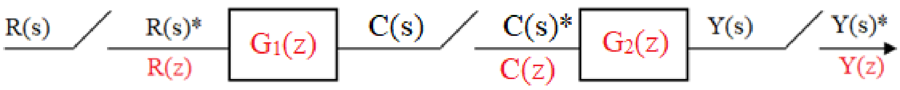


Fig. 3 Cascaded blocks in digital system

If the sampler between blocks is removed as shown in Fig. 4, then the pulse transfer function of the two blocks can be combined as:

$$\frac{Y(z)}{R(z)} = G_1G_2(z)$$

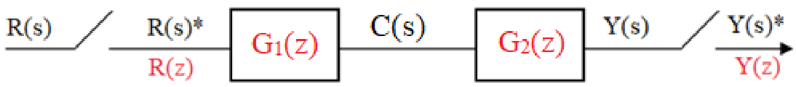


Fig. 4 Two blocks without sampler in between

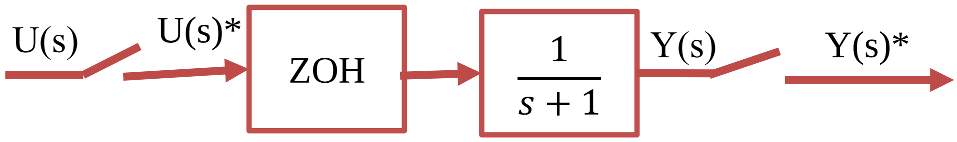
Both transfer functions are different.

$$G_1(z)G_2(z) \neq G_1G_2(z)$$

Note that the presence of samplers complicates the *algebra* of block diagrams, since the existence and expression of any input-output function depend on the number and location of the samplers.

Example:

The system 1/(S+1) is preceded by a ZOH. What will the system output be if the applied input is a unit step, assuming a sampling period of T = 1 s.



The transfer function of the ZOH is:

$$G_{ZOH}(s) = G_1(s) = \frac{1-e^{-Ts}}{s}$$

For this system, we can write:

$$Y(z) = G_{ZOH}G_2(z)u(z), \quad G_{ZOH}G_2(s) = \frac{1-e^{-Ts}}{s}\frac{1}{s+1}$$

Using partial fractional expansion we can write

$$G_{ZOH}G_2(s) = (1-e^{-Ts})\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

From Z-transform tables:

$$G_{ZOH}G_2(z) = (1-z^{-1})\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = \frac{0.63}{z-0.37}$$

Using partial fraction method, we can write

$$\frac{Y(z)}{z} = \frac{1}{z-1} - \frac{1}{z-0.37}$$

$$y(k) = 1-(0.37)^k$$

Example:

Consider the cascaded blocks with

$$H_1(s) = \frac{1}{S+2} \quad and\ H_2(s) = \frac{2}{S+4}$$

Calculate the equivalent pulse T.F in the following two cases:

a) Both blocks are connected directly,

b) Both blocks are separated by a sampler

In case (a), the equivalent pulse T.F is given as $H_1H_2(Z)$

$$H(S) = H_1(s)H_2(s) = \frac{2}{(S+2)(S+4)} = \frac{1}{S+2} - \frac{1}{S+4}$$

$$h(t) = e^{-2t} - e^{-4t}$$

$$h(kT) = e^{-2kT} - e^{-4kT} \qquad k = 0, 1, 2, 3, \ldots$$

Taking Z Transform:

$$H(Z) = \frac{Z}{Z - e^{-2T}} - \frac{Z}{Z - e^{-4T}} = \frac{Z(e^{-2T} - e^{-4T})}{(Z - e^{-2T})(Z - e^{-4T})}$$

In case (b), the equivalent pulse T.F is given as $H_1(Z)\,H_2(Z)$

$$H_1(s) = \frac{1}{S+2} \rightarrow h_1(t) = e^{-2t} \rightarrow H_1(Z) = \frac{Z}{Z - e^{-2T}}$$

$$H_2(s) = \frac{2}{S+4} \rightarrow h_2(t) = 2e^{-4t} \rightarrow H_2(Z) = \frac{2Z}{Z - e^{-4T}}$$

$$H(Z) = \frac{Z}{Z - e^{-2T}} \frac{2Z}{Z - e^{-4T}} = \frac{2Z^2}{(Z - e^{-2T})(Z - e^{-4T})}$$

Using partial fractions:

$$\frac{H(Z)}{Z} = \frac{2Z}{(Z - e^{-2T})(Z - e^{-4T})} = \frac{A}{(Z - e^{-2T})} + \frac{B}{(Z - e^{-4T})}$$

$$2Z = A(Z - e^{-4T}) + B(Z - e^{-2T})$$

At $Z = e^{-2T} \rightarrow 2e^{-2T} = A(e^{-2T} - e^{-4T}) \rightarrow A = 2e^{-2T} / (e^{-2T} - e^{-4T})$

At $Z = e^{-4T} \rightarrow 2e^{-4T} = B(e^{-4T} - e^{-2T}) \rightarrow B = 2e^{-4T} / (e^{-4T} - e^{-2T})$

$$\frac{H(Z)}{Z} = \frac{2e^{-2T}}{(e^{-2T} - e^{-4T})(Z - e^{-2T})} + \frac{2e^{-4T}}{(e^{-4T} - e^{-2T})(Z - e^{-4T})}$$

$$H(z) = \frac{2}{e^{-2T} - e^{-4T}} \left[\frac{e^{-2T} z}{z - e^{-2T}} - \frac{e^{-4T} z}{z - e^{-4T}} \right]$$

Taking inverse Z transform

$$h(kT) = \frac{2}{e^{-2T} - e^{-4T}} \left[e^{-2T} e^{-2kT} - e^{-4T} e^{-4kT}\right]$$

$$= \frac{2}{e^{-2T} - e^{-4T}} \left[e^{-2(k+1)T} - e^{-4(k+1)T}\right], \quad k = 0, 1, 2, \ldots$$

The above example clearly shows the effect of placing a sampler between analog blocks on the pulse transfer function.

Consider the control system shown in Fig. 5, the relation between input and output is given as:

$$Y(s) = R(s).G(s)$$

$$Y(s)^* = \left(RG(s)\right)^* \rightarrow Y(z) = RG(z)$$

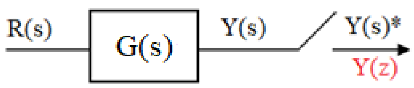


Fig. 5 open-loop system with one sampler

Therefore, the pulse T.F. is not defined, but we can calculate the system response only.

2. Pulse Transfer Function for Closed-Loop Systems

Consider the closed-loop system with samplers indicated as shown in Fig. 6

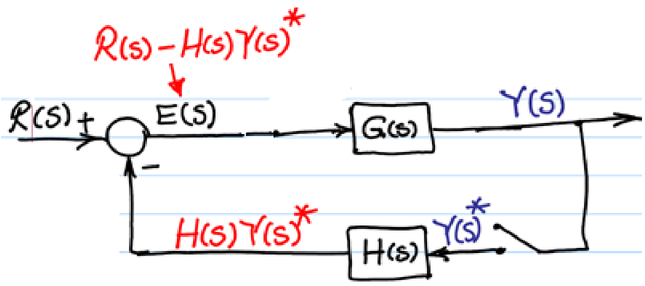


Fig. 6, closed-loop system with sampler at feedback

To obtain the pulse T.F, we assume the sampler output is Y(s)*

$$E(s) = R(s) - H(s)Y(s)^*$$

$$Y(s) = G(s)E(s)=G(s)[R(s) - H(s)Y(s)^*]$$

$$Y(s) = G(s)\,R(s) - G(s)H(s)Y(s)^*$$

$$Y(s) = G(s)\,R(s) - G(s)H(s)Y(s)^*$$

$$Y(s)^* = RG(s)^* - GH(s)^*Y(s)^*$$

$$Y(z) = RG(z) - GH(z)Y(z)$$

$$Y(z)[1 + GH(z)] = RG(z)$$

$$Y(z) = \frac{RG(z)}{1 + GH(z)}$$

It is clear that we can obtain the system response Y(z) but we can NOT obtain the pulse transfer function.

To solve this problem, a sampler must be added before the summing point as shown in Fig. 7.

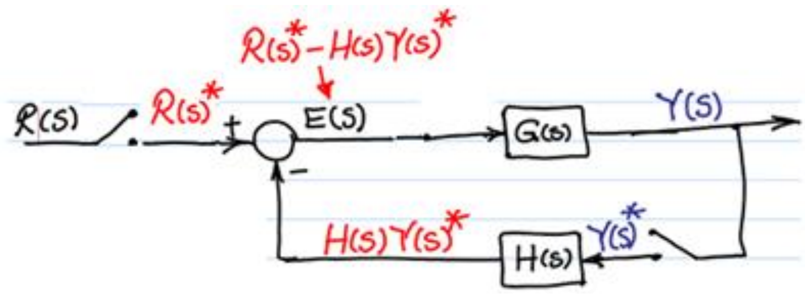


Fig. 7 closed-loop system with sampler at feedback and sampler at input

In that case the pulse T.F. is

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

Consider the discrete control system with two samplers shown in Fig. 8

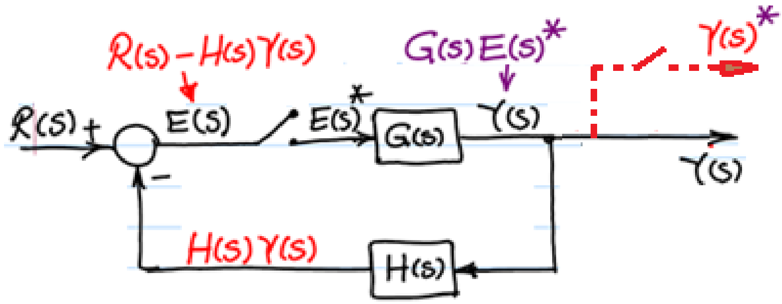


Fig. 8, closed-loop system with sampler at error signal

To obtain the pulse T.F, we assume the sampler output is E(s)*

$$E(s) = R(s) - H(s)Y(s)$$

$$E(s) = R(s) - H(s)G(s)E(s)^*$$

$$E(s)^* = R(s)^* - GH(s)^* E(s)^*$$

$$E(s)^*[1+GH(s)^*] = R(s)^*$$

$$E(s)^* = \frac{R(s)^*}{1 + GH(s)^*}$$

As the sampled error signal E(s)* multiplied by G(s)* this gives Y(s)* [Y(s)*=G(s)*E(s)*] (dashed line)

$$Y(s)^* = \frac{G(s)^* R(s)^*}{1 + GH(s)^*}$$

The pulse T.F. is given by:

$$\frac{Y(s)^*}{R(s)^*} = \frac{G(s)^*}{1 + GH(s)^*}$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

We can obtain the same pulse T.F with the configuration shown in Fig. 9

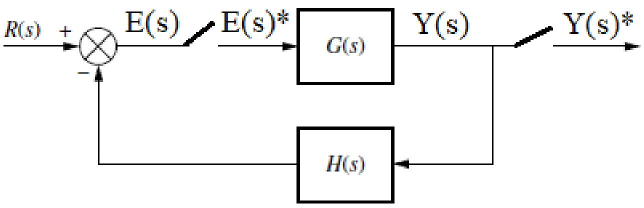
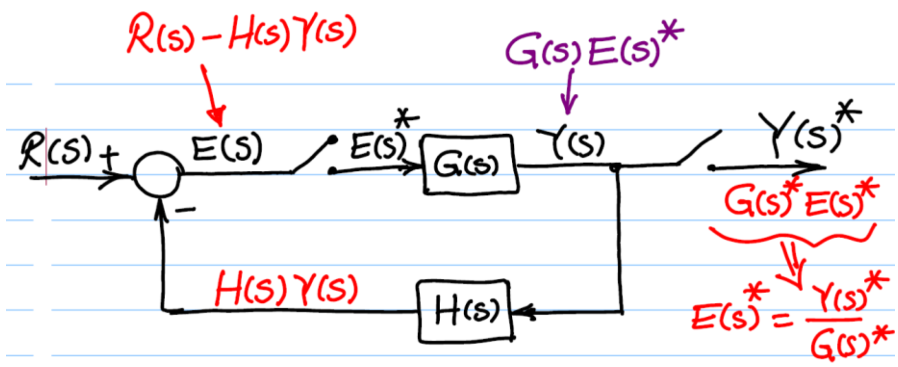


Fig. 9 closed-loop system with two samplers

To obtain the pulse T.F, we follow the steps:



Introduce a variable, with *name* E(s), at the input of the first sampler and a variable, with *name* E(s)*, at the output of that sampler. Moreover, introduce a variable, with *name* Y(s), at the input of the second sampler and a variable, with *name* Y(s)*, at the output of that sampler.

$$E(s) = R(s) - H(s)Y(s)$$

But we know that Y(s)=E(s)*G(s), by substituting by this value in above equation;

$$E(s) = R(s) - H(s)\,G(s)\,E(s)^*$$

$$E(s)^* = R(s)^* - HG(s)^*\,E(s)^*$$

$$E(s)^*[1+HG(s)^*] = R(s)^*$$

$$E(s)^* = \frac{R(s)^*}{1+GH(s)^*}$$

Replace E(s)* by Y(s)*/G(s)*

$$\frac{Y(s)^*}{G(s)^*} = \frac{R(s)^*}{1+GH(s)^*}$$

The pulse T.F. is given by:

$$\frac{Y(s)^*}{R(s)^*} = \frac{G(s)^*}{1+GH(s)^*}$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

Consider the closed-loop system with samplers indicated as shown in Fig. 10

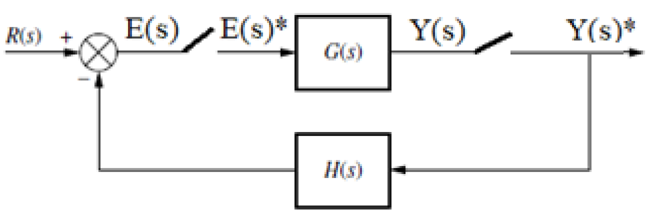
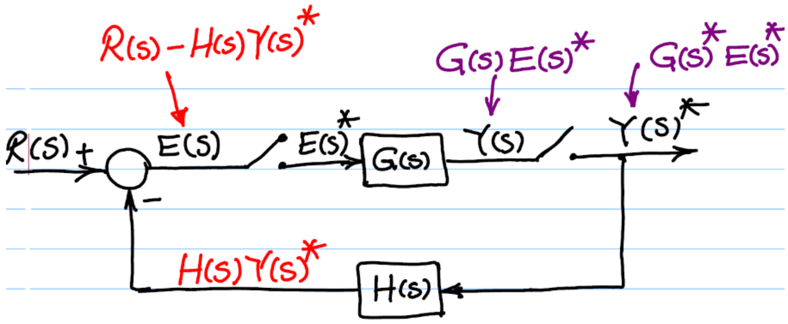


Fig. 10 closed-loop system



$$E(s) = R(s) - H(s)Y(s)^*$$

$$E(s) = R(s) - H(s)G(s)^*E(s)^*$$

$$E(s)^* = \left(R(s) - H(s)G(s)^*E(s)^*\right)^*$$

$$E(s)^* = R(s)^* - H(s)^*G(s)^*E(s)^*$$

$$E(s)^*\left(1 + G(s)^*H(s)^*\right) = R(s)^*$$

$$\frac{Y(s)^*}{G(s)^*}\left(1 + G(s)^*H(s)^*\right) = R(s)^*$$

$$\frac{Y(s)^*}{R(s)^*} = \frac{G(s)^*}{1 + G(s)^*H(s)^*}$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)}$$

We can obtain the same pulse T.F with the configuration shown in Fig. 11

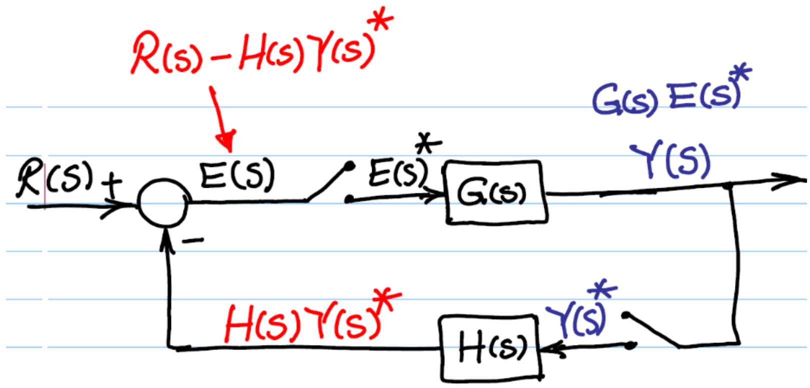
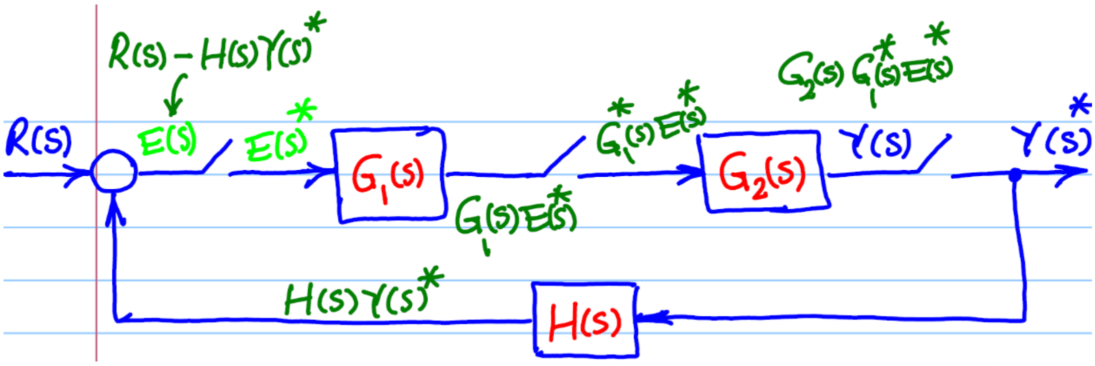


Fig. 11 Closed-loop discrete system

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)}$$

Example:



$$Y(s)^* = G_1(s)^* G_2(s)^* E(s)^* \Longrightarrow E(s)^* = \frac{Y(s)^*}{G_1(s)^* G_2(s)^*}$$

$$E(s) = R(s) - H(s)\, G_1(s)^* G_2(s)^* E(s)^*$$

$$E(s)^* = R(s)^* - H(s)^* G_1(s)^* G_2(s)^* E(s)^*$$

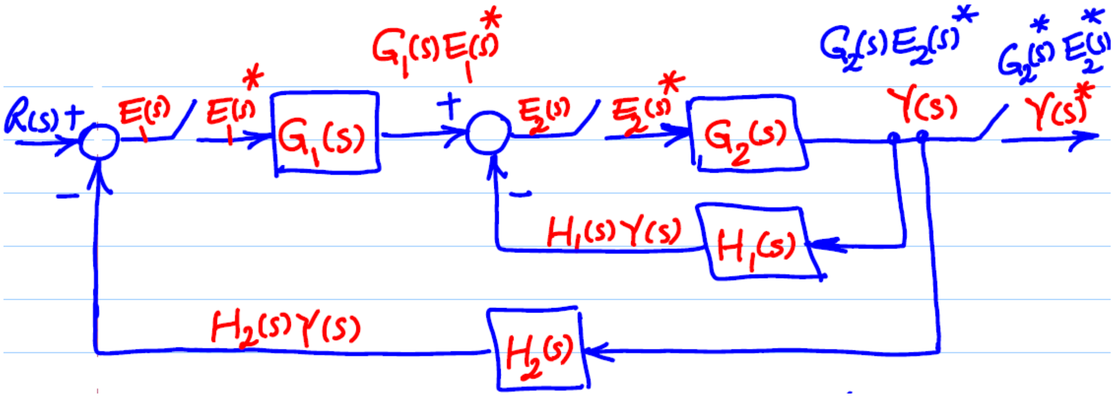
$$E(s)^* \left[1 + G_1(s)^* G_2(s)^* H(s)^*\right] = R(s)^*$$

$$\frac{Y(s)^*}{G_1(s)^* G_2(s)^*} \left[1 + G_1(s)^* G_2(s)^* H(s)^*\right] = R(s)^*$$

$$\frac{Y(s)^*}{R(s)^*} = \frac{G_1(s)^* G_2(s)^*}{1 + G_1(s)^* G_2(s)^* H(s)^*}$$

$$\frac{Y(z)}{R(z)} = \frac{G_1(z)\, G_2(z)}{1 + G_1(z)\, G_2(z)\, H(z)}$$

Example:



$$E_1(s) = R(s) - H_2(s)Y(s) \qquad Y(s) = G_2(s)E_2(s)^*$$

$$E_2(s) = G_1(s)E_1(s)^* - H_1(s)Y(s) \qquad E_2^*(s) = \frac{Y(s)^*}{G_2(s)^*}$$

$$E_1(s) = R(s) - H_2(s)G_2(s)E_2(s)^*$$

$$E_1(s)^* = R(s)^* - G_2H_2(s)^* E_2(s)^*$$

$$E_2(s) = G_1(s)R(s)^* - G_1(s)\,G_2H_2(s)^*\,E_2(s)^* - H_1(s)\,G_2(s)E_2(s)^*$$

$$E_2(s)^* = G_1(s)^*R(s)^* - G_1(s)^*\,G_2H_2(s)^*\,E_2(s)^* - G_2H_1(s)^*\,E_2(s)^*$$

$$E_2(s)^*\left[1 + G_1(s)^*\,G_2H_2(s)^* + G_2H_1(s)^*\right] = G_1(s)^*R(s)^*$$

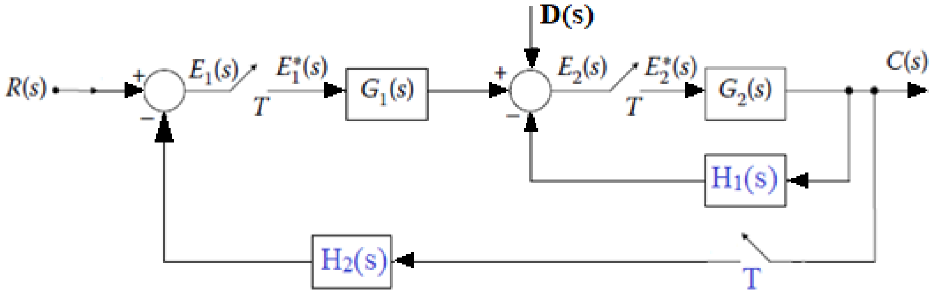
$$\frac{Y(s)^*}{G_2(s)^*}\left[1 + G_1(s)^*\,G_2H_2(s)^* + G_2H_1(s)^*\right] = G_1(s)^*R(s)^*$$

$$\frac{Y(s)^*}{R(s)^*} = \frac{G_1(s)^*\,G_2(s)^*}{1 + G_1(s)^*\,G_2H_2(s)^* + G_2H_1(s)^*}$$

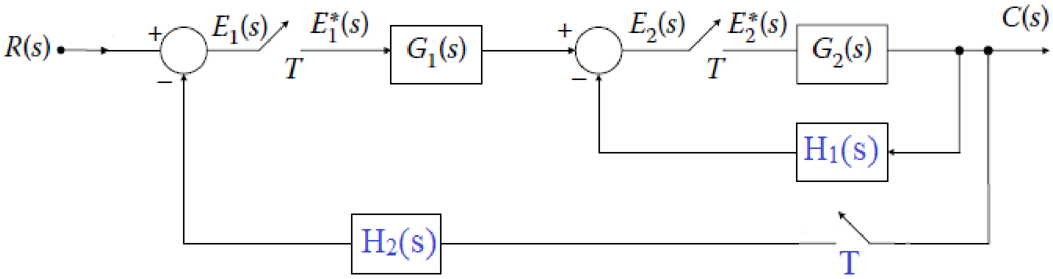
$$\frac{Y(z)}{R(z)} = \frac{G_1(z)\,G_2(z)}{1 + G_1(z)\,G_2H_2(z) + G_2H_1(z)}$$

Example:

For MISO Linear Time Invariant Digital (LTID) control system shown below, calculate the pulse transfer function.



First, we assume that D(s) = 0;



forward paths:
$P_1 = R(z)\, G_1(z)\, G_2(z)$

Individual loops:

$L_1 = -G_2 H_1(z)$
$L_2 = -G_1(z)\, G_2(z)\, H_2(z)$

The system response is:

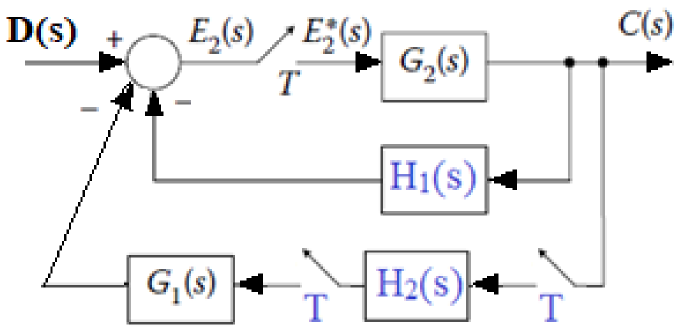
$$Y(z) = \frac{P_1}{1-(L_1+L_2)}$$

$$Y(z) = \frac{R(z)G_1(z)G_2(z)}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)}$$

The pulse T.F. is:

$$\frac{Y(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)}$$

Second, we assume that R(s) = 0;



forward paths:
$P_1 = D(z)\, G_2(z)$

Individual loops:

$L_1 = -G_2 H_1(z)$
$L_2 = -G_1(z) G_2(z) H_2(z)$

The system response is:

$$Y(z) = \frac{P_1}{1-(L_1+L_2)}$$

$$Y(z) = \frac{D(z)G_2(z)}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)}$$

The pulse T.F. is:

$$\frac{Y(z)}{D(z)} = \frac{G_2(z)}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)}$$

$$Y(z) = \frac{G_1(z)G_2(z)R(z)}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)} + \frac{G_2(z)\,D(z)}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)}$$

$$Y(z) = \frac{G_2(z)\{G_1(z)R(z)+D(z)\}}{1+G_2H_1(z)+G_1(z)G_2(z)H_2(z)}$$

Example:

Obtain the pulse T.F from the following difference equation:

$$y(k) - 0.5y(k-1) = 2x(k)$$

Taking Z-transform for both sides:

$$Y(z) - 0.5Z^{-1}Y(z) = 2X(z)$$

$$Y(z)(1-0.5Z^{-1}) = 2X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{2}{1-0.5Z^{-1}} = \frac{2Z}{Z-0.5}$$

Example:

Obtain the pulse T.F from the following difference equation:

$$y(k) + 2y(k-1) - y(k-2) = 2x(k) - x(k-1) + 2x(k-2)$$

Taking Z-transform for both sides:

$$Y(z) + 2Z^{-1}Y(z) - Z^{-2}Y(z) = 2X(z) - Z^{-1}X(z) + 2Z^{-2}X(z)$$

$$Y(z)(1+2Z^{-1}-Z^{-2}) = X(z)(2-Z^{-1}+2Z^{-2})$$

$$\frac{Y(z)}{X(z)} = \frac{2 - Z^{-1} + 2Z^{-2}}{1 + 2Z^{-1} - Z^{-2}} = \frac{2Z^2 - Z + 2}{Z^2 + 2Z - 1}$$

3. Matlab Command

The Matlab command to obtain the pulse T.F. from analog T.F. is:

```
>> g = tf(num; den);              % continuous TF
>> gd = c2d(g, T, 'zoh')          % digital TF using zero order hold and sampling time T
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3. Characteristic Equation

Characteristics equation plays an important role in the study of linear systems. As said earlier, an n^{th} order LTI discrete data system can be represented by an n^{th} order difference equation,

$$c(k + n) + a_{n-1}c(k + n - 1) + a_{n-2}c(k + n - 2) + ... + a_1c(k + 1) + a_0c(k) = b_mr(k + m) + b_{m-1}r(k + m - 1) + ... + b_0r(k)$$

where r(k) and c(k) denote input and output sequences respectively. The input output relation can be obtained by taking Z-transformation on both sides, with zero initial conditions, as

$$\frac{C(z)}{R(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}{z^n + b_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

The characteristics equation is obtained by equating the denominator of the pulse transfer function to 0, as

$$z^n + b_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

• In a causal system, the output does not precede the input. In other words, in a causal system, the output depends only on the past and present inputs, not on the future ones.

• The transfer function of a causal system is physically realizable, i.e., the system can be realized by using physical elements.

• For a causal discrete data system, the power series expansion of its transfer function must not contain any positive power in z. Positive power in z indicates prediction. Therefore, in the transfer function (given above), *n* must be greater than or equal to *m*.

m = n ⇒ proper transfer function m < n ⇒ strictly proper Transfer Function

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